SOME EXACT SOLUTIONS OF PROBLEMS OF LAMINAR JETS OF IMMISCIBLE FLUIDS

G. I. Burde

UDC 532.526

We will examine the discharge of laminar jets of fluids which are immiscible with the surrounding medium. It is assumed that there is a smooth interface between the discharge fluid and the surrounding fluid. Both fluids are assumed to be incompressible. Flow in the jet and in the external fluid is examined in a boundary-layer approximation. A problem formulated in the same manner was solved earlier by means of approximate methods: by the integral method in [1-4] and by an asymptotic method based on an expansion in powers of 1/x in [5-7].

Here, for plane and fan-shaped free and semi-infinite jets, we indicate the class of exact solutions corresponding to the case when the ratio of the absolute viscosities of the fluids is inversely proportional to the ratio of their densities. This class of solutions is extended to the case of slightly twisted fan-shaped jets.

1. Motion in the discharged and external fluids is described by the equations in the boundary-layer approximation (the quantities pertaining to the discharged fluid are designated by the subscript 1, while those pertaining to the external fluid have the subscript 2):

$$u_i\partial u_i/\partial x + v_i\partial u_i/\partial y = \nu_i\partial^2 u_i/\partial y^2, \tag{1.1}$$
$$\frac{\partial}{\partial x}(x^j u_i) + \frac{\partial}{\partial y}(x^j v_i) = 0 \quad (i = 1, 2).$$

Here and below, j = 0 for plane jets and j = 1 for fan-shaped jets.

The velocity and stress continuity conditions at the interface $y = y_*(x)$ in the same approximation are represented in the following form (due to the symmetry of the problem, we will examine only the upper half-plane):

$$u_1 = u_2, \quad \mu_1\partial u_1/\partial y = \mu_2\partial u_2/\partial y \quad \text{at} \quad y = y_*(x). \tag{1.2}$$

The conditions on the jet axis and at infinity:

$$v_1 = 0,\ \partial u_1/\partial y = 0 \text{ (free jet) at } y = 0, \tag{1.3}$$
$$v_1 = 0,\ u_1 = 0 \text{ (semi-infinite jet) at } y = 0;$$
$$u_2 \to 0 \quad \text{at} \quad y \to \infty. \tag{1.4}$$

The condition expressing the mass conservation law for the discharge fluid:

$$\int_0^{y_*(x)} u_1 dy = q, \quad q = \frac{Q}{2\rho_1} \tag{1.5}$$

(Q is a constant equal to the flow rate of the fluid in the jet). The integral relations needed to obtain a nontrivial solution to the equations are written below in the von Mises variables used to obtain the solution.

Let us change over to von Mises variables [8]: $\xi = x$, $\eta = \psi(x, y)$, where ψ is the stream function, determined by the relations $x^j u = \partial\psi/\partial y$, $x^j v = -\partial\psi/\partial x$. Instead of (1.1)-(1.5), we have an equivalent system of equalities in the variables ξ, η:

$$\frac{\partial u_i}{\partial \xi} = \nu_i \xi^{2j}\frac{\partial}{\partial \eta}\left(u_i\frac{\partial u_i}{\partial \eta}\right); \tag{1.6}$$
$$u_1 = u_2,\ \mu_1\partial u_1/\partial\eta = \mu_2\partial u_2/\partial\eta \text{ at } \eta = q; \tag{1.7}$$
$$\partial u_1/\partial\eta = 0 \text{ (free jet) at } \eta = 0, \tag{1.8}$$
$$u_1 = 0 \text{ (semi-infinite jet) at } \eta = 0.$$

Perm. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 94-98, May-June, 1988. Original article submitted February 27, 1987.

 0021-8944/88/2903-0400$12.50

Here, q is the constant value of the stream function corresponding to the interface of the fluids. It is determined in (1.5).

To convert the solution to the original variables, we use the relations

$$y = x^{-j}\int_0^{\eta}\frac{dz}{u_1(x, z)} \quad \text{at} \quad \eta \leqslant q,$$
$$y = x^{-j}\left[\int_0^{q}\frac{dz}{u_1(x, z)} + \int_q^{\eta}\frac{dz}{u_2(x, z)}\right] \quad \text{at} \quad \eta > q. \tag{1.9}$$

2. We will examine free jets. The integral condition ensuring the existence of a nontrivial solution and expressing the momentum conservation law is obtained by multiplying (1.6) by ρ_i and integrating over η: from 0 to q with i = 1 and from q to η_∞ with i = 2 [$\eta_\infty(\xi)$ is the value of the stream function corresponding to $y \to \infty$. It is found from (1.4)]. After integrating by parts, with allowance for conditions (1.7), we arrive at the relation

$$\rho_1\int_0^{q} u_1 d\eta + \rho_2\int_q^{\eta_\infty(\xi)} u_2 d\eta = \frac{J}{2} \tag{2.1}$$

(J is a constant equal to the momentum of the jet).

We will seek to find a similarity solution of Eqs. (1.6) in the form

$$u_i = \xi^m f_i(\varphi_i) \quad (\varphi_i = (\eta + b_i)/\xi^n). \tag{2.2}$$

Inserting (2.2) into (1.6), we find

$$mf_i - n\varphi_i f_i' = \nu_i (f_i f_i')' \quad (m = 2n - (2j + 1)). \tag{2.3}$$

For a free jet n = −m; the solutions of Eqs. (2.3) and the expressions for u_i have the form

$$f_i = C_i - \alpha(6\nu_i)^{-1}\varphi_i^2 \quad (\alpha = 2j + 1), \quad u_i = C_i\xi^{-\alpha/3} - \alpha(6\nu_i)^{-1}\xi^{-\alpha}(\eta + b_i)^2. \tag{2.4}$$

The constants C_1, C_2, b_1, and b_2 are determined from conditions (1.7), (1.8), and (2.1). It follows from (1.8) that $b_1 = 0$, while it follows from (1.7) that

$$C_1 = C_2, \quad q^2/\nu_1 = (q + b_2)^2/\nu_2; \tag{2.5}$$
$$\mu_1 q/\nu_1 = \mu_2(q + b_2)/\nu_2. \tag{2.6}$$

Conditions (2.5) and (2.6) are compatible if the following relation is satisfied

$$\mu_2^2/\mu_1^2 = \nu_2/\nu_1 \quad \text{or} \quad \mu_2/\mu_1 = \rho_1/\rho_2. \tag{2.7}$$

With satisfaction of (2.7), the constant b_2 is determined from (2.5) and the expressions for u_i are represented in the following form (we will henceforth omit the subscript for C)

$$u_1 = C\xi^{-\alpha/3} - \alpha(6\nu_1)^{-1}\xi^{-\alpha}\eta^2, \quad u_2 = C\xi^{-\alpha/3} - \alpha(6\nu_2)^{-1}\xi^{-\alpha}[\eta + q(\lambda - 1)]^2 \ (\lambda = \mu_2/\mu_1). \tag{2.8}$$

To calculate the constant C, we take integral condition (2.1), where the function $\eta_\infty(\xi)$, determined from (1.4), (2.8), has the form $\eta_\infty = q(1 - \lambda) + (\sqrt{6C\nu_2/\alpha})\xi^{\alpha/3}$. Using this expression in (2.1) and employing Eqs. (2.8) and Eq. (2.7), after several transformations we obtain

$$C = (3\alpha J^2/32\nu_2\rho_2^2)^{1/3}. \tag{2.9}$$

Equations (2.8) and (2.9) give the solution of the problem in von Mises variables. To change over to the variables x and y, we use (1.9).

Assuming that $\eta = q$, we use the first equation of (1.9) to obtain an expression for the half-width of the jet:

$$y_*(x) = \frac{k}{2C}x^{(j+2)/3}\ln\frac{kx^{\alpha/3} + q}{kx^{\alpha/3} - q}, \quad k = \sqrt{\frac{6C\nu_1}{\alpha}}. \tag{2.10}$$

We find y(x, η) from the first formula of (2.9) for the discharged jet ($y < y_*(x)$) and, inverting the resulting expression, we obtain

$$\eta_1 = kx^{\alpha/3}\frac{\exp(\zeta) - 1}{\exp(\zeta) + 1}, \quad \zeta = \sqrt{\frac{2C\alpha}{3\nu_1}}\frac{y}{x^{(j+2)/3}}. \tag{2.11}$$

The expression for $u_1(x, y)$ is found by inserting (2.11) into (2.8) or by using the relation $x^j u = \partial\eta/\partial y$:

$$u_1 = 4Cx^{-\alpha/3}\exp(\zeta)(\exp(\zeta) + 1)^{-2}. \tag{2.12}$$

Similarly, by using the second formulas of (1.9) and (2.8), we obtain the following at $y > y_*(x)$

$$\eta_2 = \sqrt{\frac{6C\nu_2}{\alpha}}\, x^{\alpha/3}\frac{\Phi(x)\exp(\zeta/\lambda) - 1}{\Phi(x)\exp(\zeta/\lambda) + 1} + q(1 - \lambda),$$
$$u_2 = 4Cx^{-\alpha/3}\frac{\Phi(x)\exp(\zeta/\lambda)'}{[\Phi(x)\exp(\zeta/\lambda) + 1]^2}, \quad \Phi(x) = \left(\frac{kx^{\alpha/3} + q}{kx^{\alpha/3} - q}\right)^{(\lambda-1)/\lambda}. \tag{2.13}$$

The expressions for the second component of velocity can be found from (2.11), (2.13) by means of the relation $v = -x^{-j}\partial\eta/\partial x$. Equations (2.9)-(2.13) give the solution of problem (1.1)-(1.5) with condition (2.7). This can be proven by direct substitution. The asymptotic formula for the solution at $x \gg 1$ can be compared with the results in [5] for the case $j = 0$ (plane jet); the respective expressions agree, given condition (2.7).

3. We will examine semi-infinite jets. We obtain the integral condition which ensures a nontrivial solution by assuming that Eq. (2.7) is satisfied. We multiply (1.6), with $i = 1$, by $\rho_1^2\eta$. With $i = 2$, we multiply (1.6) by $\rho_2^2(\eta + b_2)$ [b_2 is found from Eq. (2.6)]. We then integrate within the respective limits and add the resulting equalities. With allowance for (1.7), after we take the relations $\rho_1^2\nu_1 = \rho_2^2\nu_2$, $\mu_2 q = \mu_1(q + b_2)$ – which follow from (2.6), (2.7) – and we integrate them by parts as well, we have the integral condition in the form

$$\rho_1^2\int_0^q \eta u_1 d\eta + \rho_2^2\int_q^{\eta_\infty(\xi)}(\eta + b_2)u_2 d\eta = \frac{E}{2} \quad (E \text{ is a constant}). \tag{3.1}$$

Similarity solution (2.2) is determined by Eq. (2.3) with the relation which imposes condition (3.1): $2n = -m$, from which $m = -\alpha/2$, $n = \alpha/4$. The solution of Eq. (2.3):

$$f_i = C_i\varphi_i^{1/2} - \alpha(6\nu_i)^{-1}\varphi_i^2.$$

The conditions at the interface (1.7) lead to the relations [$b_1 = 0$ due to (1.8)] $C_1 q^{1/2} = C_2(q + b_2)^{1/2}$, $q^2/\nu_1 = (q + b_2)^2/\nu_2$, $\mu_1 C_1 q^{-1/2} = \mu_2 C_2(q + b_2)^{-1/2}$, $\mu_1 q/\nu_1 = \mu_2(q + b_2)/\nu_2$, which are compatible with (2.7). The expressions for u_1, u_2 have the form

$$u_1 = C_2\sqrt{\lambda}\xi^{-5/8\alpha}\eta^{1/2} - \alpha(6\nu_1)^{-1}\xi^{-\alpha}\eta^2,$$
$$u_2 = C_2\xi^{-5/8\alpha}(\eta + b_2)^{1/2} - \alpha(6\nu_2)^{-1}\xi^{-\alpha}(\eta + b_2)^2, \quad b_2 = q(\lambda - 1), \tag{3.2}$$
$$C_2 = (10E/3\rho_2^2)^{3/8}(\alpha/6\nu_2)^{5/8}$$

[the formula for C_2 was obtained from condition (3.1)].

Using (1.9) and (3.2) to transform the solution to the original variables, we obtain the expressions

$$y_*(x) = \frac{A}{3C_2\sqrt{\lambda}}x^{(2j+3)/4}\left[\ln\frac{F^2 + F + 1}{(F - 1)^2} + 2\sqrt{3}\,\mathrm{arctg}\frac{2F + 1}{\sqrt{3}}\right],$$
$$y \leqslant y_*:\ \eta_1 = A^2x^{\alpha/4}z^2, \quad u_1 = C_2\sqrt{\lambda}Ax^{-\alpha/2}z(1 - z^3),$$
$$y = \frac{2A}{C_2\sqrt{\lambda}}x^{(2j+3)/4}\int_0^z\frac{dt}{1 - t^3};$$
$$y > y_*:\ \eta_2 = \lambda A^2x^{\alpha/4}s^2 + q(1 - \lambda), \quad u_2 = C_2\sqrt{\lambda}Ax^{-\alpha/2}s(1 - s^3),$$
$$y = \frac{2A}{C_2\sqrt{\lambda}}x^{(2j+3)/4}\left[\int_0^{F(x)}\frac{dt}{1 - t^3} + \lambda\int_{F(x)}^s\frac{dt}{1 - t^3}\right], \quad F = (\sqrt{q}/A)x^{-\alpha/8},$$
$$A = (6\nu_1C_2\sqrt{\lambda}/\alpha)^{1/3},$$

which give the relations $\eta_i(x, y)$ and $u_i(x, y)$ in parametric form.

4. We will examine swirled fan-shaped jets. In the "weak swirling" approximation, the equations for the azimuthal component of velocity w_i are separated, while integral relations

(2.1), (3.1) remain as before. These conditions must be augmented by an integral relation which ensures nonambiguity of the solution of the equations for the azimuthal component. We will write the equations for w_i and the corresponding boundary conditions in von Mises variables:

$$\frac{\partial w_i}{\partial \xi} + \frac{w_i}{\xi} = \nu_i \xi^2 \frac{\partial}{\partial \eta}\left(u_i \frac{\partial w_i}{\partial \eta}\right); \tag{4.1}$$

$$w_1 = w_2, \ \mu_1 \partial w_1/\partial \eta = \mu_2 \partial w_2/\partial \eta \ \text{ at } \ \eta = q, \ \partial w_1/\partial \eta = 0 \tag{4.2}$$

(free jet) at $\eta = 0$, $w_1 = 0$ (semi-infinite jet) at $\eta = 0$,

$$w_2 = 0 \ \text{ at } \ \eta = \eta_\infty.$$

To derive the integral relation, we multiply (4.1) by ρ_i for the free jet and by $\rho_i^2(\eta + b_i)$ for the semi-infinite jet. We then integrate within the corresponding limits and add. In subsequent transformations, we use conditions (4.2). In the case of a semi-infinite jet, we also use relations (2.6), (2.7) and the additional proposition $w_i(\xi, \eta) = \sigma(\xi) u_i(\xi, \eta)$. The validity of the latter is confirmed by the form of the resulting solution. As a consequence, $\rho_1 \xi \int_0^q w_1 d\eta + \rho_2 \xi \int_q^{\eta_\infty(\xi)} w_2 d\eta = L$ (free jet), $\rho_1^2 \xi \int_0^q \eta w_1 d\eta + \rho_2^2 \xi \int_q^{\eta_\infty(\xi)} (\eta + b_2) w_2 d\eta = M$ (semi-infinite jet).

We seek the solutions of Eqs. (4.1) in the form $w_i = \xi^l G_i(\varphi_i)$, $\varphi_i = (\eta + b_i)/\xi^n$. Omitting the details of the calculations, we write the final expressions for the azimuthal component of velocity: $w_i = (2L/J) x^{-1} u_i(x, y)$ (free jet), $w_i = (2M/E) x^{-1} u_i(x, y)$ (semi-infinite jet).

LITERATURE CITED

1. A. L. Genkin, V. I. Kukes, and L. P. Yarin, "Propagation of a jet of immiscible fluids," Probl. Teploenerg. Prikl. Teplofiz., 9 (1973).
2. H. Yu. and G. F. Scheele, "Laminar jet contraction and velocity distribution in immiscible liquid–liquid systems," Int. J. Multiphase Flow, 2, No. 2 (1975).
3. I. P. Penchev, S. P. Radev, and R. K. Rakadjiev, "Velocity profile relaxation of a jet in a liquid–liquid system," Theoretical and Applied Mechanics: 3rd International Conference, Varna, 1977. Bulgarian Academy of Sciences, Sophia (1977), Vol. 1.
4. M. M. Anwar, A. Bright, et al., "Laminar liquid jets in immiscible liquid systems," Trans. Inst. Chem. Eng., 60, No. 5 (1982).
5. V. I. Eliseev, "Discharge of laminar jets of immiscible incompressible fluids," Gidroaeromekh. Teor. Uprugosti, 21 (1976).
6. V. I. Eliseev, "Asymptotic solution of the problem of the discharge of radial laminar jets of immiscible fluids," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1977).
7. V. I. Eliseev, L. I. Sukhikh, and L. A. Fleer, "Asymptotic method of solving the problem of the discharge of radial laminar jets of immiscible fluids," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3 (1980).
8. L. G. Loitsyanskii, Laminar Boundary Layer [in Russian], Fizmatgiz, Moscow (1962).